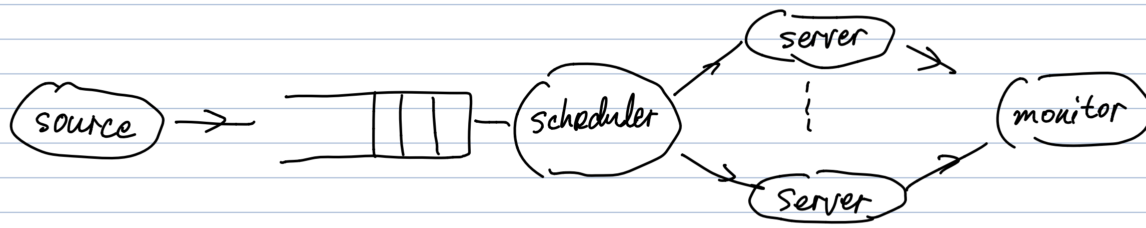
Lecture 14.

Age-optimal Scheduling in Queues.



Packet i. generation time. arrival time. delivery time

S_i C_i D_i

$$0 \le S_1 \le S_2 \le \cdots$$

$$S_i \le C_i \le D_i.$$

S_i and C_i are arbitrarily given.

In-order arrivals: $S_i \le S_{i+1}$. $C_i \le C_{i+1}$.

Out-of-order arrivals: $S_i \le S_{i+1}$. $C_i > C_{i+1}$,

B: buffer size.

M: No. of servers.

If $B=0$, system can keep M packets,

i.i.d. expenential service times,

AoI:

$$\Delta(t) = t - \max\{S_i : D_i \le t\}.$$

Scheduler determines:

① which packet to send by servers over time.

②. packet dropping & replacing.

- causal / non-anticipative policies:
  scheduling decisions made based on history & current state of the system.

- preemptive policies:
  server can switch to another packet any time.

- non-preemptive policies:
  server must complete current packet before serving another.

- work-conserving policies:
  all servers are busy when some packets are waiting in the queue.

Π: all causal policies.

Good scheduling policies for minimizing AoI:

- Last-Generated, First-Served (LGFS),
  latest generated packet is served first, with ties broken arbitrarily.
- Last-Come, First-Served (LCFS),
  last arrived packet is served first, with ties broken arbitrarily.

for in-order arrivals, LGFS = LCFS.

Conjecture:

LGFS minimizes AoI.

How to show it?

Stochastic orders:

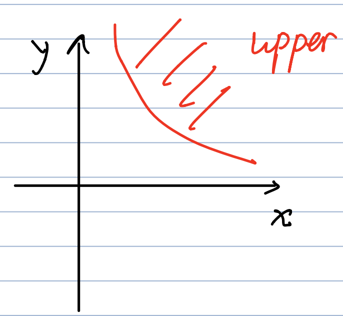
Def: X is stochastically smaller Y, denoted by $X \leq_{st} Y$, if

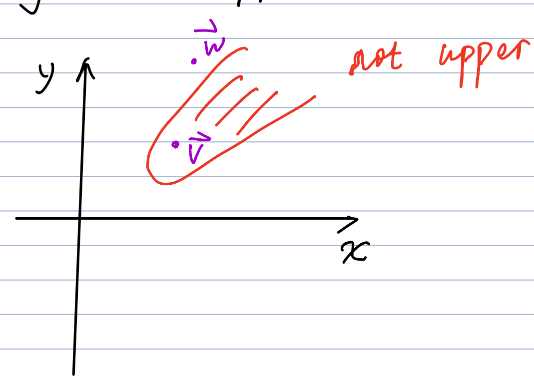
$$\Pr\{X > x\} \leq \Pr\{Y < x\}, \quad \forall x \in \mathbb{R}.$$

Def: A set U is called upper, if $\vec{y} \in U$ whenever $\vec{x} \in U$, $\vec{y} \geq \vec{x}$.

$[x, \infty) = \{a : a \geq x\}$ is upper.

$\{(x, y) : x + y \geq 1\}$ is upper.





Def: $\vec{X} \leq_{st} \vec{Y}$, $\vec{X}$ is stochastically smaller than $\vec{Y}$, if

$$\Pr\{\vec{X} \in U\} \leq \Pr\{\vec{Y} \in U\}, \quad \forall \text{ upper } U \in \mathbb{R}^n.$$

Def: $\{X(t), t \geq 0\} \leq_{st} \{Y(t), t \geq 0\}$, $\{X(t), t \geq 0\}$ is stochastically small than $\{Y(t), t \geq 0\}$, if for $\forall n$, $\forall\ 0 \leq t_1 < t_2 < \cdots < t_n$.

$$(X(t_1), X(t_2), \cdots, X(t_n)) \leq_{st} (Y(t_1), Y(t_2), \cdots, Y(t_n)).$$

Lemma 1: $X \leq_{st} Y$, if & only if there exist X_1, Y_1 in the same probability space $\{\Omega, \mathcal{F}, P)$, such that

$X =_{st} X_1$.

$Y =_{st} Y_1$.

$X_1 \leq Y_1$ with prob 1.

$Pr\{X_1 \leq Y_1\} = 1$.

Lemma 2. $X \leq_{st} Y$, if & only if for all non-decreasing function $p(\cdot)$,

$E[p(X)] \leq E[p(Y)]$.

whenever expectiation exist,

Reading: Schocastic Orders.

Def: functional.

a mapping from functions $\{ P_1(\cdot), P_1(\cdot), \cdots \}$ to real numbers in $\mathbb{R}$.

A function p can be viewed as a vector

$P(\cdot)$, $\{ p(x_1), p(x_2), \cdots \}$ is a vector.

Hence, a functional is a mapping from vectors to $\mathbb{R}$.

Lemma 3.

$\{X(t), t \geq 0\} \leq_{st} \{Y(t), t \geq 0\}$ if & only if

for all non-decreasing functionals f.

$E[f(\{X(t), t \geq 0\})] \leq E[f(\{Y(t), t \geq 0\})]$.

whenever expectations exist.

Def: non-decreasing functional f.

A functional f is non-decreasing if

$f(\{x(t), t \geq 0\}) \leq f(\{y(t), t \geq 0\})$ whenever

$x(t) \leq y(t)$. $\forall t \geq 0$.

Examples:

non-decreasing functionals of AoI process:

① Time-average age:

$$f_1(\{\Delta(t), t \geq 0\}) = \frac{1}{T}\int_0^T \Delta(t)\,dt.$$

② Time-average age penalty function:

$$f_2(\{\Delta(t), t \geq 0\}) = \frac{1}{T}\int_0^T P(\Delta(t))\,dt.$$

P is non-decreasing function.

③. $\Pr\{\Delta \geq d\}$:

indicator function:

$$1\{\Delta(t) \geq d\} = \begin{cases} 1 & \text{if } \Delta(t) \geq d \\ 0 & \text{if } \Delta(t) < d, \end{cases}$$

$$\Pr\{\Delta \geq d\} = \frac{1}{T}\int_0^1 1\{\Delta(t) \geq d\}\,dt.$$

Theorem:

If service times are i.i.d. across servers & time. then. for all No. of servers $M \geq 1$, all buffer size $B \geq 0$, all generation/arrival times. $I = \{S_i, C_i\}$. all policies $\pi \in \Pi$.

$$[\{\Delta_{prmp-LGFS}(t), t \geq 0\} \mid I]$$

$$\leq_{st} [\{\Delta_\pi(t), t \geq 0\} \mid I],$$

or for all non-decreasing functionals f

$$E[f(\{\Delta_{prmp-LGFS}(t), t \geq 0\}) \mid I]$$

$$\leq E[f(\{\Delta_\pi(t), t \geq 0\}) \mid I],$$

provided expectations exist.

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preemptive LGFS is age-optimal.